VCU, Department of Computer Science

CMSC 302

Sets

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Introduction to Set Theory (§2.1)

- A *set* is a new type of structure, representing an *unordered* collection (group, plurality) of zero or more *distinct* (different) objects.
- Set theory deals with operations between, relations among, and statements about sets.
- Sets are ubiquitous (universal) in computer software systems.
- *All* of mathematics can be defined in terms of some form of set theory (using predicate logic).

Basic notations for sets

- For sets, we'll use variables *S*, *T*, *U*, …
- We can denote a set *S* in writing by *listing* all of its elements in curly braces:
  - {a, b, c} is the set of whatever 3 objects are denoted by a, b, c.
- *Set builder notation*: For any proposition *P*(*x*) over any universe of discourse, {*x*|*P*(*x*)} **is *the set*** *of all x such that P*(*x*).

Basic properties of sets

- Sets are inherently *unordered*:
  - No matter what objects a, b, and c denote, {a, b, c} = {a, c, b} = {b, a, c} = {b, c, a} = {c, a, b} = {c, b, a}.
- All elements are *distinct* (unequal); multiple listings make no difference!
  - If a=b, then {a, b, c} = {a, c} = {b, c} = {a, a, b, a, b, c, c, c, c}.
  - This set contains (at most) 2 elements!

Definition of Set Equality

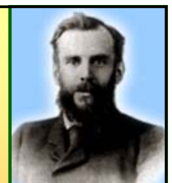
- Two sets are declared to be *equal if and only if* they contain exactly the same elements.
- *In particular, it does not matter how the set is defined or denoted.*
- **For example:**
  The set {1, 2, 3, 4}
  = {x | x is an integer where x>0 and x<5 } =
  {x | x is a positive integer whose square is >0 and <25}

Infinite Sets

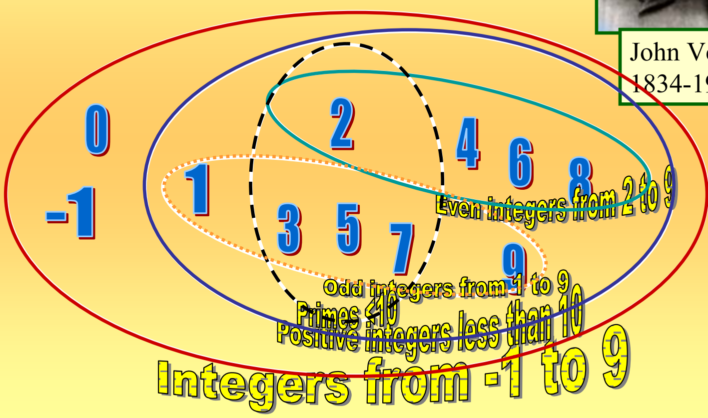
- Conceptually, sets may be *infinite* (*i.e.,* not *finite*, without end, unending).
- Symbols for some special infinite sets:
  N = {0, 1, 2, …} The **N**atural numbers.
  Z = {…, -2, -1, 0, 1, 2, …} The **Z**ntegers.
  R = The "**R**eal" numbers, such as 374.1828471929498181917281943125…
- "Blackboard Bold" or double-struck font ($\mathbb{N}$,$\mathbb{Z}$,$\mathbb{R}$) is also often used for these special number sets.
- Infinite sets come in different sizes!

More on this after module #4 (functions).

Venn Diagrams



John Venn
1834-1923



Basic Set Relations: Member of

- Def. $x \in S$ ("x is in S") is the proposition that object x is an *∈lement* or *member* of set S.
  - *e.g.* $3 \in$ **N**, "a"$\in${x | x is a letter of the alphabet}
  - Can define set equality in terms of $\in$ relation:
    $\forall S,T: S=T \leftrightarrow (\forall x: x \in S \leftrightarrow x \in T)$
    "Two sets are equal iff they have all the same members."
- $x \notin S :\equiv \neg(x \in S)$ "x is not in S"

The Empty Set

- Def. $\varnothing$ ("null", "the empty set") is the unique set that contains no elements whatsoever.
- $\varnothing = \{\} = \{x|\textbf{False}\}$
- No matter the domain of discourse, we have:
- Axiom. $\neg\exists x\colon x\in\varnothing$.

Subset and Superset Relations

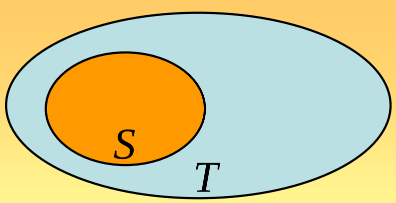
- Def. $S\subseteq T$ ("S is a **subset** of T", also pronounced ***S is contained in T***) means that every element of S is also an element of T.
- $S\subseteq T \Leftrightarrow \forall x\,(x\in S \rightarrow x\in T)$
- $\varnothing\subseteq S$, $S\subseteq S$.

- Def. $S\supseteq T$ ("S is a **superset** of T", also pronounced ***S includes T***) means $T\subseteq S$.
- Note $S{=}T \Leftrightarrow S\subseteq T \wedge S\supseteq T$.
- $S\nsubseteq T$ means $\neg(S\subseteq T)$, *i.e.* $\exists x\,(x\in S \wedge x\notin T)$

Proper (Strict) Subsets & Supersets

- Def. $S\subset T$ ("S is a proper subset of T") means that $S\subseteq T$ but $T\nsubseteq S$.

i.e. there exists at least one element of T not contained in *S*



Example:
$\{1, 2\} \subset \{1, 2, 3\}$

Venn Diagram equivalent of $S\subset T$

Proper (Strict) Subsets & Supersets

Example:

Consider a set {a, b, c ,d ,e}.

Then sets {d, b, a}, {c, e}, {e} and $\varnothing$ are proper subsets

but, {a, b, f}, {k} and {e, b, a ,d ,c} **are not**!

Sets Are Objects, Too!

- The objects that are elements of a set may *themselves* be sets.
- *E.g.* let $S = \{x \mid x \subseteq \{1,2,3\}\}$
  then $S = \{\varnothing,$
  $\{1\}, \{2\}, \{3\},$
  $\{1,2\}, \{1,3\}, \{2,3\},$
  $\{1,2,3\}$
- $\}$
- Note that $1 \neq \{1\} \neq \{\{1\}\}$!!!!

Very Important!

Cardinality and Finiteness

- Def. $|S|$ (read "the *cardinality* of *S*") is a measure of how many different elements *S* has.
- *E.g.*, $|\varnothing|=0$, $|\{1,2,3\}| = 3$, $|\{a,b\}| = 2$,
  $|\{\{1,2,3\},\{4,5\}\}| =$ 2
- If $|S| \in \mathbb{N}$, then we say *S* is *finite*. Otherwise, we say *S* is *infinite*.
- What are some infinite sets we've seen?
- $\mathbb{N}, \mathbb{Z}, \mathbb{R}$

The *Power Set* Operation

- Def. The *power set* P(*S*) of a set *S* is the set of all subsets of *S*. $P(S) :\equiv \{x \mid x \subseteq S\}$.
- *E.g.* $P(\{a,b\}) = \{\varnothing, \{a\}, \{b\}, \{a,b\}\}$.
- Sometimes P(*S*) is written 2^S.
- Remark. For finite *S*, $|P(S)| = 2^{|S|}$.
- It turns out $\forall S{:}|P(S)|>|S|$, *e.g.* $|P(\mathbb{N})| > |\mathbb{N}|$.
- ***There are different sizes of infinite sets!***

Ordered *n*-tuples

- These are like sets, except that duplicates matter, and the order makes a difference.
- Def. For $n \in \mathbf{N}$, an *ordered n-tuple* or a *sequence* or *list of length n* is written $(a_1, a_2, \ldots, a_n)$. Its *first* element is a_1, *etc.*
- Note that $(1, 2) \neq (2, 1) \neq (2, 1, 1)$. ← Contrast with sets' {}
- Empty sequence, singlets, pairs, triples, quadruples, quintuples, …, *n*-tuples.

Cartesian Products

- Def. For sets A, B, their *Cartesian product* $A \times B :\equiv \{(a, b) \mid a \in A \wedge b \in B\}$.
- *E.g.* $\{a,b\} \times \{1,2\} = \{(a,1),(a,2),(b,1),(b,2)\}$
- Remarks (AxB is a set of ORDERED *n-tuples*)
- For finite A, B, $|A \times B| = |A||B|$.
- The Cartesian product is *not* commutative: *i.e.*, $\neg\forall AB: A \times B = B \times A$.
- Extends to $A_1 \times A_2 \times \ldots \times A_n$...

René Descartes (1596-1650)

Review: Set Notations So Far

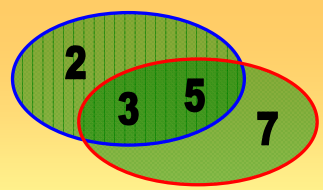
- Variable objects x, y, z; sets S, T, U.
- Literal set {a, b, c} and set-builder $\{x \mid P(x)\}$.
- $\in$ relational operator, and the empty set $\varnothing$.
- Set relations $=, \subseteq, \supseteq, \subset, \supset, \not\subset$, etc.
- Venn diagrams.
- Cardinality $|S|$ and infinite sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$.
- Power sets P(S).

Start §2.2: The Union Operator

- Def. For sets A, B, their *union* $A \cup B$ is the **SET** containing all elements that are either in A, **or** ("$\vee$") in B (or, of course, in both).
- Formally, $\forall A,B: A \cup B = \{x \mid x \in A \vee x \in B\}$.
- Remark. $A \cup B$ is a **superset** of both A and B (in fact, it is the smallest such superset): $\forall A, B: (A \cup B \supseteq A) \wedge (A \cup B \supseteq B)$

Union Examples

- $\{a,b,c\} \cup \{2,3\} = \{a,b,c,2,3\}$ Required Form
- $\{2,3,5\} \cup \{3,5,7\} = \{2,3,5,3,5,7\} = \{2,3,5,7\}$



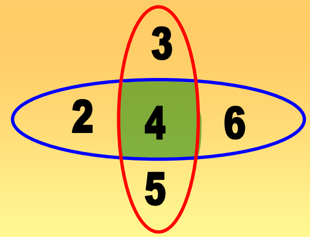
Think "The **Uni**ted States of America includes every person who worked in <u>any</u> U.S. state last year." (This is how the IRS sees it...)

The Intersection Operator

- Def. For sets A, B, their *intersection* $A \cap B$ is the set containing all elements that are simultaneously in A **and** ("$\wedge$") in B.
- Formally, $\forall A,B$: $A \cap B = \{x \mid x \in A \wedge x \in B\}$.
- Remark. $A \cap B$ is a **subset** of both A and B (in fact it is the largest such subset): $\forall A, B$: $(A \cap B \subseteq A) \wedge (A \cap B \subseteq B)$

Intersection Examples

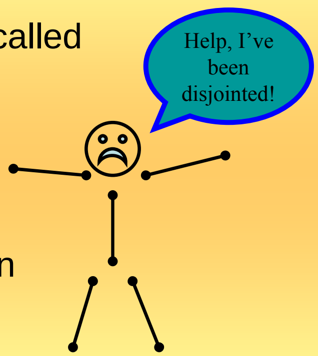
- $\{a,b,c\} \cap \{2,3\}$ = $\varnothing$
- $\{2,4,6\} \cap \{3,4,5\}$ = $\{4\}$



Think "The **intersection** of W Franklin St. and Jefferson St. is just that part of the road surface that lies on ***both*** streets."

Disjointedness

- Def. Two sets A, B are called *disjoint* (*i.e.*, unjoined) iff their intersection is empty. ($A \cap B = \varnothing$)
- Example: the set of even integers is disjoint with the set of odd integers.



Inclusion-Exclusion Principle

- How many elements are in $A \cup B$?
  $|A \cup B| = |A| + |B| - |A \cap B|$
- Example: How many students are on our class email list? Consider set $E = I \cup M$,
  $I = \{s \mid s$ turned in an information sheet$\}$
  $M = \{s \mid s$ sent the TAs their email address$\}$
- Some students did both!
  $|E| = |I \cup M| = |I| + |M| - |I \cap M|$

Subtract out items in intersection, to compensate for double-counting them!

Set Difference

- Def. For sets A, B, the *difference of A and B*, written $A-B$, is the set of all elements that are in A but not B.
- Formally:

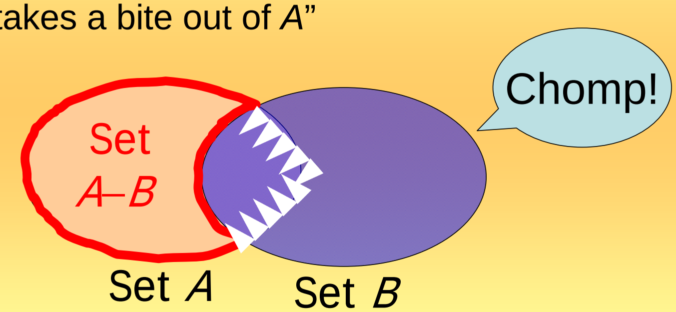
  $$A - B :\equiv \{x \mid x\in A \wedge x\notin B\}$$
  $$= \{x \mid \neg(x\in A \rightarrow x\in B) \}$$
- Also called:
  The *complement of B with respect to A*.

Set Difference Examples

- $\{1,2,3,4,5,6\} - \{2,3,5,7,9,11\} =$ $\{1,4,6\}$
- $\mathbb{Z} - \mathbb{N} = \{\ldots, -1, 0, 1, 2, \ldots\} - \{0, 1, \ldots\}$
  $= \{x \mid x \text{ is an integer but not a nat. \#}\}$
  $= \{x \mid x \text{ is a negative integer}\}$
  $= \{\ldots, -3, -2, -1\}$

Set Difference - Venn Diagram

- $A-B$ is what's left after B "takes a bite out of A"



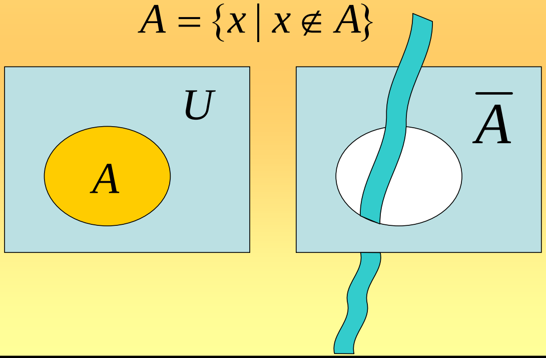
Set Complements

- Def. The *universe of discourse* can itself be considered a set, call it U.
- When the context clearly defines U, we say that for any set $A\subseteq U$, the *complement* of A, written $\overline{A}$, is the complement of A w.r.t. U, *i.e.*, it is $U-A$.
- *E.g.*, If U=**N**, $\overline{\{3,5\}} = \{0,1,2,4,6,7,\ldots\}$

More on Set Complements

- An equivalent definition, when *U* is clear:

$$\overline{A} = \{x \mid x \notin A\}$$



Set Identities

- Identity: $A \cup \varnothing = A = A \cap U$
- Domination: $A \cup U = U$, $A \cap \varnothing = \varnothing$
- Idempotent: $A \cup A = A = A \cap A$
- Double complement: $(\overline{\overline{A}}) = A$
- Commutative: $A \cup B = B \cup A$, $A \cap B = B \cap A$
- Associative: $A \cup (B \cup C) = (A \cup B) \cup C$,
  $A \cap (B \cap C) = (A \cap B) \cap C$

DeMorgan's Law for Sets

- Exactly analogous to (and provable from) DeMorgan's Law for propositions.

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Proving Set Identities

- To prove statements about sets, of the form
  $E_1 = E_2$ (where the *E*s are set expressions), here are three useful techniques:
- 1. Prove $E_1 \subseteq E_2$ and $E_2 \subseteq E_1$ separately.
- 2. Use set builder notation & logical equivalences.
- 3. Use a *membership table*.

Method 1: Mutual subsets

- Example:
- Show $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$.
- Part 1: Show $A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap C)$.
  - Assume $x\in A\cap(B\cup C)$, & show $x\in(A\cap B)\cup(A\cap C)$.
  - We know that $x\in A$, and either $x\in B$ or $x\in C$.
    - Case 1: $x\in B$. Then $x\in A\cap B$, so $x\in(A\cap B)\cup(A\cap C)$.
    - Case 2: $x\in C$. Then $x\in A\cap C$, so $x\in(A\cap B)\cup(A\cap C)$.
  - Therefore, $x\in(A\cap B)\cup(A\cap C)$.
  - Therefore, $A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap C)$.
- Part 2: Show $(A\cap B)\cup(A\cap C)\subseteq A\cap(B\cup C)$. …

Method 2: Use set builder notation & logical equivalences

- Show $\overline{A\cap B}=\overline{A}\cup\overline{B}$

See Ex.11, page 125 in edition 6 of our textbook

Method 3: Membership Tables

- Just like truth tables for propositional logic.
- Columns for different set expressions.
- Rows for all combinations of memberships in constituent sets.
- Use "**1**" to indicate membership in the derived set, "**0**" for non-membership.
  (trick is, use **MAX for** $\cup$, and **min for** $\cap$)
- Prove equivalence with identical columns.

Membership Table Example

- Prove $(A\cup B)-B = A-B$.

Hint: think about an element *x* which does or doesn't belong to A and/or B

$A-B :\equiv \{x \mid x\in A \wedge x\notin B\}$

A	B	$A\cup B$	$(A\cup B)-B$	$A-B$
0	0	0		0
0	1	1		0
1	0	1		1
1	1	1		0

Membership Table Exercise

- Prove $(A \cup B) - C = (A - C) \cup (B - C)$.

A	B	C	$A \cup B$	$(A \cup B) - C$	$A - C$	$B - C$	$(A - C) \cup (B - C)$
0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0
0	1	0	1	1	0	1	1
0	1	1	1	0	0	0	0
1	0	0	1	1	1	0	1
1	0	1	1	0	0	0	0
1	1	0	1	1	1	1	1
1	1	1	1	0	0	0	0

Generalized Unions & Intersections

- Since union & intersection are commutative and associative, we can extend them from operating on *ordered pairs* of sets (A,B) to operating on sequences of sets $(A_1,\ldots,A_n)$, or even on unordered *sets* of sets,

$$X=\{A \mid P(A)\}.$$

Generalized Union

- Binary union operator:
- $A \cup B$
- *n*-ary union:
  $A \cup A_2 \cup \ldots \cup A_n :\equiv ((\ldots((A_1 \cup A_2) \cup \ldots) \cup A_n)$
  (grouping & order is irrelevant)
- "Big U" notation: $\bigcup_{i=1}^{n} A_i$
- or for infinite sets of sets: $\bigcup_{A \in X} A$

Generalized Intersection

- Binary intersection operator:
- $A \cap B$
- *n*-ary intersection:
  $A_1 \cap A_2 \cap \ldots \cap A_n \equiv ((\ldots((A_1 \cap A_2) \cap \ldots) \cap A_n)$
  (grouping & order is irrelevant)
- "Big Arch" notation: $\bigcap_{i=1}^{n} A_i$
- or for infinite sets of sets: $\bigcap_{A \in X} A$

Representations

- A frequent theme of this course will be methods of *representing* one discrete structure using another discrete structure of a different type.
- *E.g.*, one can represent natural numbers as
  - Sets: **0**:≡∅, **1**:≡{**0**}, **2**:≡{**0**,**1**}, **3**:≡{**0**,**1**,**2**}, …
  - Bit strings: **0**:≡`0`, **1**:≡`1`, **2**:≡`10`, **3**:≡`11`, **4**:≡`100`, …

Representing Sets with Bit Strings

- For an enumerable u.d. U with ordering $x_1, x_2, \ldots$, represent a finite set $S \subseteq U$ as the finite bit string $B = b_1 b_2 \ldots b_n$ where $\forall i: x_i \in S \leftrightarrow (i<n \wedge b_i=1)$.
- *E.g.* $U=\mathbf{N}$, $S=\{2,3,5,7,11\}$,
  B=01101010001.

Review: Set Operations § 2.2

- Union
- Intersection
- Set difference
- Set complements
- Set identities
- Set equality proof techniques:
  - Mutual subsets.
  - Derivation using logical equivalences.
- Set representations

References

- Rosen
  Discrete Mathematics and its Applications, 6e
  Mc GrawHill, 2007